

Transient growth calculations obtained directly from the Orr–Sommerfeld matrices

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We develop a method for computing the optimal disturbance based on the Orr–Sommerfeld–Squire matrices. The method is similar to the one employed elsewhere in the literature. The basic method compares well when compared to a benchmark case in single-phase flow. In contrast, for two-phase flows, the basic method needs to be modified in a substantial manner before agreement can be obtained with known test cases. These modifications are discussed and derived below and eventually, good agreement between the present case and the two-phase test cases is obtained.

I. BACKGROUND

The Orr–Sommerfeld–Squire equation for a generic single-phase problem in the hydrodynamic stability of a parallel flow can be written down in generic form as follows:

$$\mathcal{L}\chi = \lambda\mathcal{M}\chi, \quad (1)$$

The stability problem is solved at a particular set of wavenumbers (α, β) , and the Orr–Sommerfeld–Squire matrices \mathcal{L} and \mathcal{M} and the eigenvalue λ all depend on the wavenumbers. The extension to the two-phase case is carried out below in Section III. Recall, the state vector χ is obtained by writing the wall-normal velocity and vorticity in a finite Chebyshev approximation:

$$w(z) = \sum_{i=0}^n A_i T_i(x), \quad \eta = \sum_{i=0}^n B_i T_i(x), \quad x = 2z - 1,$$

such that

$$\chi = (A_0, \dots, A_n, B_0, \dots, B_n)^T.$$

In our previous work [1], we have demonstrated how the matrices in Equation (1) can be used to solve the corresponding initial-value problem. The initial-value problem is formulated as follows:

$$\frac{\partial}{\partial t} \mathcal{M}\chi = \mathcal{L}\chi, \quad t > 0, \quad (2a)$$

with initial condition

$$\chi(t=0) = \chi_0, \quad \chi_0 = (x_0, \dots, x_n, y_0, \dots, y_n)^T, \quad (2b)$$

and where

$$x_0 = \frac{1}{\pi} \int_{-1}^1 T_0(x) w\left(\frac{1}{2}(x+1), t=0\right) \frac{dx}{\sqrt{1-x^2}}, \quad (3a)$$

$$x_i = \frac{2}{\pi} \int_{-1}^1 T_i(x) w\left(\frac{1}{2}(x+1), t=0\right) \frac{dx}{\sqrt{1-x^2}}, \quad i \neq 0, \quad (3b)$$

$$y_0 = \frac{1}{\pi} \int_{-1}^1 T_0(x) \eta\left(\frac{1}{2}(x+1), t=0\right) \frac{dx}{\sqrt{1-x^2}}, \quad (3c)$$

$$y_i = \frac{2}{\pi} \int_{-1}^1 T_i(x) \eta\left(\frac{1}{2}(x+1), t=0\right) \frac{dx}{\sqrt{1-x^2}}, \quad i \neq 0. \quad (3d)$$

The evolution operator

In Reference [1], we have shown how the solution to Equation (2) can be written as

$$\chi(t_p) = \mathcal{E}_{t_p} \chi_0, \quad t_p = p\Delta t, \quad p = 0, 1, \dots,$$

where $\Delta t \rightarrow 0$, keeping $t_p = t$ finite. Also \mathcal{E}_t is the **evolution operator**, where

$$\mathcal{E}_t = \lim_{\Delta t \rightarrow 0} [(\mathcal{M} - \Delta t \mathcal{L})^{-1} \mathcal{M}]^p. \quad (4)$$

Note that Equation (4) amounts to solving the linear differential algebraic equation (DAE) (2a) using the backward Euler method. Previously (e.g. in Reference [1]) we used a trapezoidal rule. However, in the present calculations, we have found by trial and error that the backward Euler method is the most accurate one for our purposes, and a particular advantage of the Backward Euler method is its large domain of stability.

Alternative derivation of the evolution operator

An obvious solution to Equation (2) (yet equivalent to the one in Equation (4)) is

$$\chi(t) = \sum_{q=1}^N \mu_q e^{\lambda_q t} \chi_{(q)}, \quad (5a)$$

where $(\chi_{(q)}, \lambda_q)$ are an eigenvector-eigenvalue pair:

$$\mathcal{L} \chi_{(q)} = \lambda_{(q)} \mathcal{M} \chi_{(q)}, \quad q = 1, 2, \dots, N, \quad (5b)$$

where $N = 2(n+1)$ for the problem in Equation (2). This solution is a complete solution provided the eigenvectors form a complete basis, that is, the matrix

$$V = \begin{pmatrix} | & & | \\ \chi_{(1)} & \cdots & \chi_{(N)} \\ | & & | \end{pmatrix}, \quad (5c)$$

is invertible. We work with the assumption that V is indeed invertible, but that $V^\dagger V$ is not a diagonal matrix because the eigenvalue problem is non-normal. The problem here is to relate the μ_q -coefficients to the initial data in Equation (2b).

We define

$$\chi_{i0} = \langle \mathbf{e}_i, \boldsymbol{\chi}_0 \rangle, \quad \chi_i(t) = \langle \mathbf{e}_i, \boldsymbol{\chi}(t) \rangle$$

where \mathbf{e}_i denotes the usual basis in \mathbb{C}^N , $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on the same, and $\boldsymbol{\chi}_0 = (\chi_{01}, \dots, \chi_{0N})^T$. From Equation (5a) we have

$$\begin{aligned} \chi_{i0} &= \langle \mathbf{e}_i, \boldsymbol{\chi}_0 \rangle, \\ &= \sum_q \mu_q \langle \mathbf{e}_i, \boldsymbol{\chi}_{(q)} \rangle, \\ &= \sum_q \mu_q V_{qi}, \end{aligned}$$

hence

$$(\mu_1, \dots, \mu_N)^T = V^{-1} \boldsymbol{\chi}_0.$$

Also from Equation (5a) we have

$$\begin{aligned} \chi_i(t) &= \langle \mathbf{e}_i, \boldsymbol{\chi}(t) \rangle, \\ &= \sum_q \mu_q e^{\lambda_q t} \langle \mathbf{e}_i, \boldsymbol{\chi}_{(q)} \rangle, \\ &= \sum_q \mu_q e^{\lambda_q t} V_{iq}, \\ &= \sum_q (V^{-1} \boldsymbol{\chi}_0)_q e^{\lambda_q t} V_{iq}, \\ &= \sum_q \sum_j V_{iq} e^{\lambda_q t} V_{qj}^{-1} \chi_{0j}, \\ &= \sum_j (V e^{\Lambda t} V^{-1})_{ij} \chi_{0j}, \end{aligned}$$

hence

$$\boldsymbol{\chi}(t) = (V e^{\Lambda t} V^{-1}) \boldsymbol{\chi}_0,$$

and

$$\mathcal{E}_t = V e^{\Lambda t} V^{-1}, \tag{6}$$

where

$$e^{\Lambda t} = \text{diag} (e^{\lambda_{(1)} t}, \dots, e^{\lambda_{(N)} t}),$$

It therefore follows from these calculations and from Equation (4) that

$$\mathcal{E}_t = \lim_{\Delta t \rightarrow 0} [(\mathcal{M} - \Delta t \mathcal{L})^{-1} \mathcal{M}]^p = V e^{\Lambda t} V^{-1} \tag{7}$$

where again the limit $\Delta t \rightarrow 0$ is taken in Equation (7), keeping $t_p = p\Delta t = t$ finite. Throughout this work, both methods have been used with identical results, although using the eigenvalue/eigenvector method has proved more expedient in certain situations.

II. THE BASIC METHOD

The idea of the transient-growth calculation is to start with the energy norm

$$E(t) = \frac{1}{2k^2} \int_0^1 (|\partial_z w|^2 + k^2|w|^2 + |\eta|^2) dz, \quad k^2 = \alpha^2 + \beta^2, \quad (8)$$

and at each point in time to optimize the energy norm subject to the constraint that $E(0) = 1$. The resulting maximum energy is called the *maximum amplification factor*, $G(t)$. These calculations can be done within the framework of Section I as follows. First, the energy norm in Equation (8) is identified with a scalar product on the space of admissible χ -vectors:

$$\begin{aligned} E(t) &= \frac{1}{2} \frac{1}{2k^2} \int_{-1}^1 dx \left[\left| \sum_{i=0}^n A_i e_x T'_i(x) \right|^2 + k^2 \left| \sum_{i=0}^n A_i T_i(x) \right|^2 + \left| \sum_{i=0}^n B_i T_i(x) \right|^2 \right], \quad e_x = \frac{dx}{dz} = 2, \\ &= \frac{1}{2k^2} \sum_{i,j} A_i^* A_j \left(\frac{1}{2} e_x^2 \int_{-1}^1 T'_i(x) T'_j(x) dx \right) + \frac{1}{2k^2} \sum_{i,j} A_i^* A_j \left(\frac{1}{2} k^2 \int_{-1}^1 T_i(x) T_j(x) dx \right) \\ &\quad + \frac{1}{2k^2} \sum_{i,j} B_i^* B_j \left(\frac{1}{2} \int_{-1}^1 T_i(x) T_j(x) dx \right). \end{aligned}$$

Call

$$\begin{aligned} \mathbb{T}_{ij}^{(1)} &= \frac{1}{2} e_x^2 \int_{-1}^1 T'_i(x) T'_j(x) dx, \\ \mathbb{T}_{ij}^{(0)} &= \frac{1}{2} \int_{-1}^1 T_i(x) T_j(x) dx. \end{aligned}$$

We have

$$\begin{aligned} E(t) &= \frac{1}{2k^2} \left[\sum_{i,j} A_i^* A_j \mathbb{T}_{ij}^{(1)} + k^2 \sum_{i,j} A_i^* A_j \mathbb{T}_{ij}^{(0)} + \sum_{i,j} B_i^* B_j \mathbb{T}_{ij}^{(0)} \right], \\ &= \frac{1}{2k^2} \left\langle \chi, \begin{pmatrix} \mathbb{T}^{(1)} + k^2 \mathbb{T}^{(0)} & 0 \\ 0 & \mathbb{T}^{(0)} \end{pmatrix} \chi \right\rangle, \\ &:= \frac{1}{2k^2} \langle \chi, \mathbb{T} \chi \rangle. \end{aligned}$$

Here, the angle brackets denote the usual scalar product on the space of χ -vectors, and the matrix \mathbb{T} is symmetric positive-definite. Thus, the equation

$$E(t) = \frac{1}{2k^2} \langle \chi, \mathbb{T} \chi \rangle$$

defines a scalar product on the space of χ -vectors. However, we have

$$\chi = \mathcal{E}_t \chi_0,$$

hence

$$E(t) = \frac{1}{2k^2} \langle \chi_0, \mathcal{E}_t^\dagger \mathbb{T} \mathcal{E}_t \chi_0 \rangle.$$

Thus, the optimization to be performed can be recast as an optimization of the functional

$$E[\chi_0] = \frac{1}{2k^2} \langle \chi_0, \mathcal{E}_t^\dagger \mathbb{T} \mathcal{E}_t \chi_0 \rangle,$$

subject to the constraint that

$$\frac{1}{2k^2} \langle \chi_0, \mathbb{T} \chi_0 \rangle = 1.$$

In other words, we have the following Lagrange-multiplier problem:

$$E[\chi_0] = \frac{1}{2k^2} \langle \chi_0, \mathcal{E}_t^\dagger \mathbb{T} \mathcal{E}_t \chi_0 \rangle - \lambda \left(\frac{1}{2k^2} \langle \chi_0, \mathbb{T} \chi_0 \rangle - 1 \right).$$

The optimum vector is obtained by setting

$$\frac{\delta E}{\delta \chi_0^*} = 0,$$

in other words,

$$\mathcal{E}_t^\dagger \mathbb{T} \mathcal{E}_t \chi_0 = \lambda \mathbb{T} \chi_0. \quad (9)$$

Equation (9) is a generalized eigenvalue problem, and it is readily checked that the eigenvalues are real (both matrices appearing in the problem are Hermitian) and moreover, that the eigenvalues are non-negative. It can be further shown by a straightforward calculation (backsubstitution into the constrained functional) that

$$\sup_{\chi_0} \left[E[\chi_0] - \lambda \left(\frac{1}{2k^2} \langle \chi_0, \mathbb{T} \chi_0 \rangle - 1 \right) \right] = \max \lambda,$$

where the maximum is taken over the spectrum of the generalized eigenvalue problem (9). Thus, at each point in time, the maximum amplification factor is

$$G(t) = \max \lambda.$$

Validation – single-phase flow

We have validated this procedure against the known test case of Poiseuille flow. We work in the units used by Orzag and other later researchers for their stability calculations of single-phase Poiseuille flow [2]. Thus, we take $\alpha = 1$, $\beta = 0$, and two cases for the Reynolds number: $Re = 5000$ (asymptotically stable) and $Re = 8000$ (asymptotically unstable). A comparison between known results for $G(t)$ in this instance and the results from our own calculations is shown in Figure 1.

It is now of interest to examine the behavior demonstrated in Figure 1 a little further. We go back over to our own units based on the full channel height and the friction velocity and examine the features of the transient growth in the supercritical case $Re = 8000$, $Re_* = \sqrt{8 \times 8000} \approx 252.9822$, for various times $t \in [0, 2]$ (corresponding to times $[0, 2] Re_*/2$ in Figure 1). The resulting study is presented in Figure 1 where it should be noticed that it is the square root of the energy of the most-amplified disturbance that is plotted in a wavenumber space, for different t -values. For very short times ($t = 0.1$) the transiently most-amplified mode has a wavevector with components in both the streamwise and spanwise

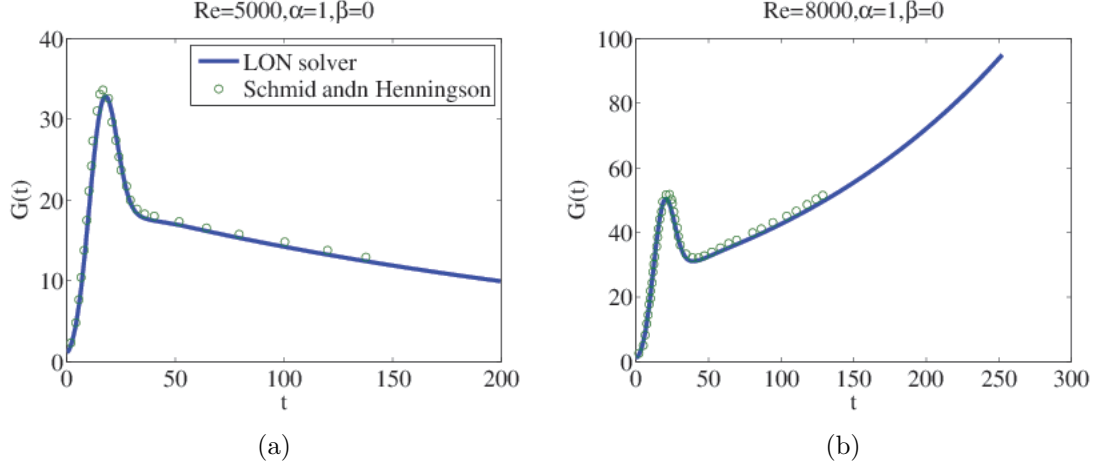


FIG. 1. Validation of our code for the maximum amplification factor compared to known benchmark case in the literature (data from Reference [3]). The small discrepancies between the two datasets are due to errors in scanning and digitizing the data from the reference text.

directions (at $t = 0.1$, $\max_{\alpha,\beta} G_{\alpha,\beta}(t)$ occurs at $(\alpha, \beta) \approx (3, 8)$). As time goes by, the most-amplified mode moves to a more spanwise wavenumber such that by $t = 1$ the maximum value $\max_{\alpha,\beta} G_{\alpha,\beta}(t)$ occurs at $\alpha \approx 0$ and $\beta = 6$. Thereafter, there is a slow evolution of the trajectory of the most-amplified disturbance through the wavenumber space away from spanwise wavenumbers towards streamwise ones (the eigenvalue theory predicts that as $t \rightarrow \infty$ the most-amplified disturbance is a streamwise-only mode – Figure 3). By $t = 10$ the asymptotic state is reached and the most-amplified disturbance is indeed streamwise-only.

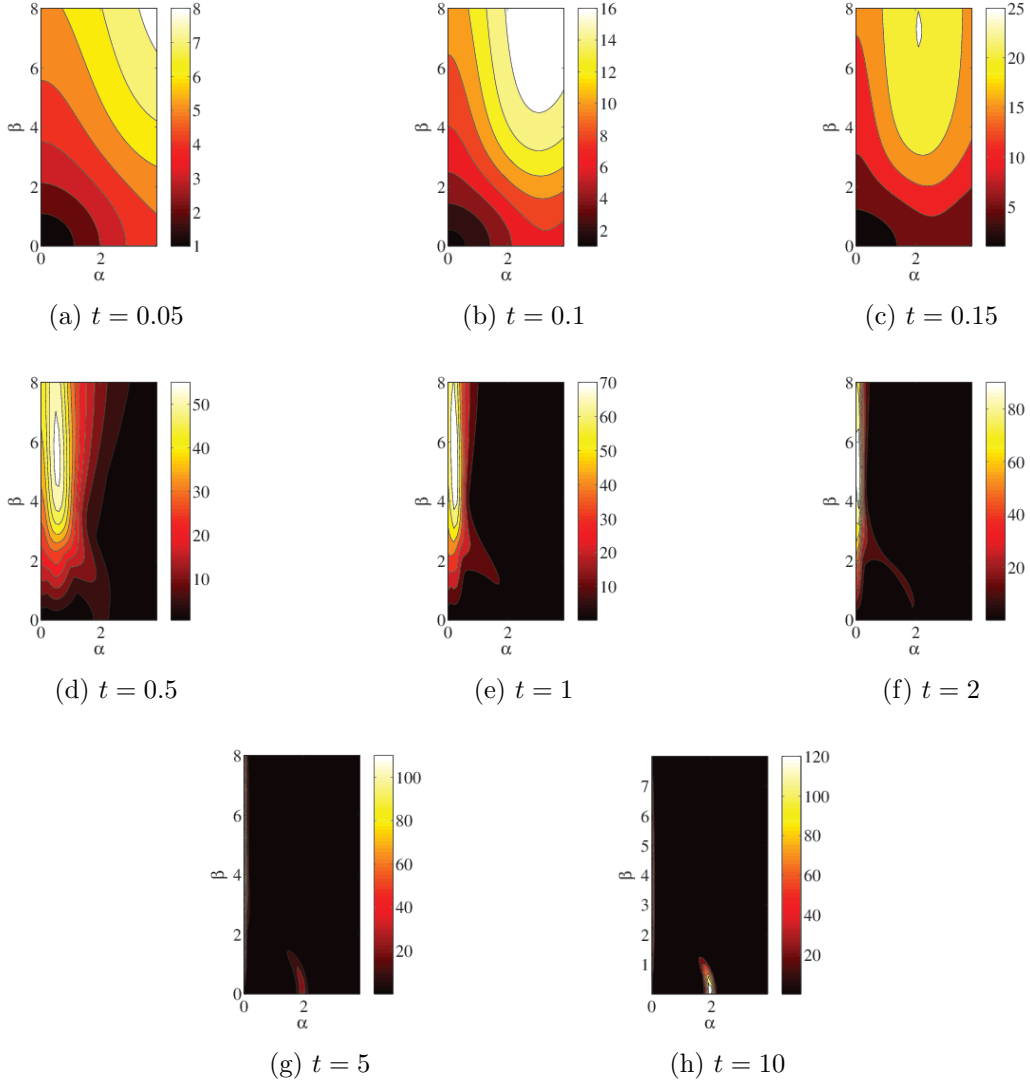


FIG. 2. Time evolution of the maximum amplification factor as a function of the wavenumbers α (streamwise) and β (spanwise). Between $t = 0.1$ and $t = 10$ the optimal disturbance moves from being spanwise-dominated to streamwise-dominated.

III. TWO-PHASE FLOWS

Application of the basic method described above to two-phase flows with surface tension has produced significant quantitative differences between the present method and the existing test cases in the literature (e.g. Reference [4]), albeit that the qualitative trends are the same. We have investigated this, and the discrepancy can be reduced (in many cases eliminated entirely) by projecting out the most *stable* eigenmodes from the transient-growth calculations. The methodology for doing this is described below.

First, in more detail, the evolution operator $\mathcal{E}_t = V e^{At} V^{-1}$ involves all eigenmodes that arise in the truncated Chebyshev expansion of the full problem. It is well known that only the most unstable eigenmodes are computed accurately in the Chebyshev collocation method. Therefore, contributions to the evolution operator coming from highly *stable* modes

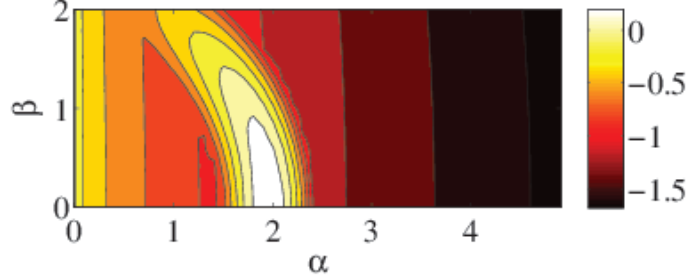


FIG. 3. Single-phase flow eigenvalue analysis, corresponding to $t \rightarrow \infty$ and to be read in conjunction with Figure 2: Eigenvalue of most-dangerous mode of the Orr–Sommerfeld–Squire equations, with $Re = 5000$. The most-dangerous mode according to eigenvalue analysis (valid as $t \rightarrow \infty$) is a streamwise one.

will lead to numerical error. Although such stable modes are of no interest an an eigemode analysis (or more generally, in any kind of analysis wherein the limit $t \rightarrow \infty$ is taken), they could interfere with transient-growth calculations at finite time. Therefore, as in the standard literature on transient growth, the proposal here is to project such modes out of the evolution operator.

To do this, an approximate solution to the initial value problem (2a) is proposed involving only the first Q modes:

$$\chi(t) = \sum_{q=1}^Q \kappa_q(t) \chi_{(q)}, \quad \chi_{(q)} = (w_{L(q)}, w_{G(q)}, \eta_{L(q)}, \eta_{G(q)})^T, \quad Q \leq N, \quad (10)$$

where L and G label the phases. The solution (10) is substituted into our calculation for the energy norm, which for a two-phase flow experiencing surface tension but no gravity, reads

$$E(t) = \frac{1}{2k^2} \left\{ \left[\sum_{L,G} \int (|\partial_z w|^2 + k^2 |w|^2 + |\eta|^2) dz \right] + \frac{k^4}{We} |f_0|^2 \right\}, \quad (11)$$

where We is the Weber number and f_0 represents the interface disturbance. In order to do this calculation, we need for example expressions such as the following, valid for either one of the phases:

$$\int |w|^2 dz = \int \left| \sum_q \kappa_q w_q(z) \right|^2 dz,$$

which can be done in the Chebyshev expansion as follows:

$$\begin{aligned}
\int |w|^2 dz &= \int \left| \sum_q \kappa_q w_q(z) \right|^2 dz, \\
&= \int \left| \sum_q \kappa_q \sum_j A_{j(q)} T_j(x) \right|^2 dz, \\
&= \sum_q \sum_{q'} \kappa_q \kappa_{q'}^* \sum_j \sum_{j'} A_{j(q)} \mathbb{T}_{jj'}^{(0)} A_{j'(q')}^*.
\end{aligned}$$

Similarly,

$$\int |\eta|^2 dz = \sum_q \sum_{q'} \kappa_q \kappa_{q'}^* \sum_j \sum_{j'} B_{j(q)} \mathbb{T}_{jj'}^{(0)} B_{j'(q')}^*,$$

for either phase. In this way, it is clear that

$$\begin{aligned}
E(t) &= \frac{1}{2k^2} \sum_{q,q'} \kappa_q \kappa_{q'}^* \left[\frac{k^4}{We} f_{0(q)} f_{0(q')}^* \right] \\
&+ \frac{1}{2k^2} \sum_{q,q'} \kappa_q \kappa_{q'}^* \left[\sum_{L,G} \left(\sum_{jj'} A_{j(q)} \mathbb{T}_{jj'}^{(1)} A_{j'(q')}^* + k^2 \sum_{jj'} A_{j(q)} \mathbb{T}_{jj'}^{(0)} A_{j'(q')}^* + \sum_{jj'} B_{j(q)} \mathbb{T}_{jj'}^{(0)} B_{j'(q')}^* \right) \right].
\end{aligned} \tag{12}$$

Clearly, this is a grotesque expression but it can be ameliorated. Let us momentarily revert to an eigenvalue with a single component, with $[V, D] = \text{eig}(\mathcal{L}, \mathcal{M})$ say, where the q^{th} column of the matrix V corresponds to the q^{th} eigenvector, with $V_{(q)} = (A_{0(q)}, \dots, A_{N(q)})^T$ (say). Then, for any $N \times N$ symmetric matrix C , we have

$$\begin{aligned}
(V^\dagger C V)_{q'q} &= \sum_{jj'} V_{q'j'}^\dagger C_{j'j} V_{jq}, \\
&= \sum_{jj'} V_{j'q'}^* C_{j'j} V_{jq}, \\
&= \sum_{jj'} A_{j'(q')}^* C_{j'j} A_{j(q)}, \\
&= \sum_{jj'} A_{j'(q')}^* C_{jj'} A_{j(q)}
\end{aligned} \tag{13}$$

Hence, by introducing the matrix

$$\mathbb{T} = \begin{pmatrix} \mathbb{T}_L^{(1)} + k^2 \mathbb{T}_L^{(0)} & & & & \\ & \mathbb{T}_G^{(1)} + k^2 \mathbb{T}_G^{(0)} & & & \\ & & \mathbb{T}_L^{(0)} & & \\ & & & \mathbb{T}_G^{(0)} & \\ & & & & k^4/We \end{pmatrix}$$

it should be clear from Equation (13) that Equation (11) can be rewritten as

$$E(t) = \frac{1}{2k^2} \sum_{qq'} \kappa_q \kappa_{q'}^* \left(V_Q^\dagger \mathbb{T} V_Q \right)_{q'q}, \tag{14}$$

where now V_Q is an $N \times Q$ matrix where each column is an eigenvector of the full problem, where $N = 2(N_L + N_G + 2) + 1$; in other words, the q^{th} column of V_Q is the eigenvector

$$V_{(q)} = (A_{0(q)}^L, \dots, A_{N_L(q)}^L, A_{0(q)}^G, \dots, A_{N_G(q)}^G, B_{0(q)}^L, \dots, B_{N_L(q)}^L, B_{0(q)}^G, \dots, B_{N_G(q)}^G)^T.$$

We consider again Equation (14). Because the κ_q 's are weights in an eigenfunction expansion of the solution of a linear evolutionary equation, we have $\kappa_q = e^{\lambda_q t} \mu_q$, where μ_q is a constant. Thus,

$$\begin{aligned} E(t) &= \frac{1}{2k^2} \sum_{qq'} \kappa_q \kappa_{q'}^* \left(V_Q^\dagger \mathbb{T} V_Q \right)_{q'q}, \\ &= \frac{1}{2k^2} \sum_{qq'} \mu_{q'}^* e^{\lambda_{q'}^* t} \left(V_Q^\dagger \mathbb{T} V_Q \right)_{q'q} e^{\lambda_q t} \mu_q, \\ &= \frac{1}{2k^2} \sum_{qq'} \mu_{q'}^* \left(e^{\Lambda^* t} V_Q^\dagger \mathbb{T} V_Q e^{\Lambda t} \right)_{q'q} \mu_q, \\ &:= \frac{1}{2k^2} \langle \vec{\mu}, \left(e^{\Lambda^* t} V_Q^\dagger \mathbb{T} V_Q e^{\Lambda t} \right) \vec{\mu} \rangle, \end{aligned}$$

where the last line makes use of an obvious notation. The relation

$$E_Q(t) = \frac{1}{2k^2} \langle \vec{\mu}, \left(e^{\Lambda_Q^* t} V_Q^\dagger \mathbb{T} V_Q e^{\Lambda_Q t} \right) \vec{\mu} \rangle \quad (15)$$

is our final expression for the energy of a disturbance involving only the first Q eigenmodes: a subscript Q has been introduced (with $E(t) \rightarrow E_Q(t)$) to remind ourselves that only Q eigenmodes are used in the expansions.

We carry out several consistency checks on our calculations. The first one involves checking that the matrix multiplication $V_Q^\dagger \mathbb{T} V_Q$ makes sense. We first of all note the size of the various matrices: V_Q is a $N \times Q$ matrix and \mathbb{T} is a matrix of size $N \times N$. Hence, doing the ‘cross multiplication’ check to see if the product $V_Q^\dagger \mathbb{T} V_Q$ is defined, we have

$$(Q \times N) \times (N \times N) \times (N \times Q) = (Q \times Q),$$

and the matrix multiplication is therefore consistent. Next, we would also like to check that $E_{Q=N}(t)$ agrees with our earlier expressions for the energy, recalled here from Section II as

$$\begin{aligned} E(t) &= \frac{1}{2k^2} \langle \vec{x}, \mathcal{E}^\dagger \mathbb{T} \mathcal{E} \vec{x} \rangle, \\ &= \frac{1}{2k^2} \langle \vec{x}, V^{-1\dagger} e^{\Lambda^* t} V^\dagger \mathbb{T} V e^{\Lambda t} V^{-1} \vec{x} \rangle, \\ &= \frac{1}{2k^2} \langle V^{-1} \vec{x}, \left(e^{\Lambda^* t} V^\dagger \mathbb{T} V e^{\Lambda t} \right) (V^{-1} \vec{x}) \rangle, \end{aligned} \quad (16)$$

Clearly, if we set $\vec{\mu} = V^{-1} \vec{x}$ in Equation (16) and $Q = N$ in Equation (15) we have

$$E_{N=Q} = E(t) = \frac{1}{2k^2} \langle V^{-1} \vec{x}, \left(e^{\Lambda^* t} V^\dagger \mathbb{T} V e^{\Lambda t} \right) (V^{-1} \vec{x}) \rangle = \frac{1}{2k^2} \langle \vec{\mu}, \left(e^{\Lambda^* t} V^\dagger \mathbb{T} V e^{\Lambda t} \right) \vec{\mu} \rangle,$$

and the two approaches are identical (and hence consistent) when $N = Q$. However, even when $Q < N$, we can identify $\vec{x} = V_Q \vec{\mu}$ in Equation (15), which will be helpful in what follows.

We obtain the maximum growth $G(t)$ in the usual way by maximizing the constrained problem

$$E_Q[\vec{\mu}] = \frac{1}{2k^2} \langle \vec{\mu}, \left(e^{\Lambda_Q^* t} V_Q^\dagger \mathbb{T} V_Q e^{\Lambda_Q t} \right) \vec{\mu} \rangle - \lambda \left[\frac{1}{2k^2} \langle \vec{\mu}, \left(V_Q^\dagger \mathbb{T} V_Q \right) \vec{\mu} \rangle - 1 \right],$$

As before, the maximum growth rate is obtained by setting

$$\frac{\delta E_Q}{\delta \vec{\mu}^*} = 0,$$

hence

$$\left(e^{\Lambda_Q^* t} V_Q^\dagger \mathbb{T} V_Q e^{\Lambda_Q t} \right) \vec{\mu} = \lambda \left(V_Q^\dagger \mathbb{T} V_Q \right) \vec{\mu}$$

and hence

$$\sup_{\vec{\mu}} E_Q[\vec{\mu}] = \max \lambda = G(t).$$

The optimal vector is the corresponding eigenvector of the problem. In the usual basis, the optimal vector is given by the transformation $\vec{x} = V_Q \vec{\mu}$.

Validation

We have validated these calculations with respect to Figures 4 and 5 in Reference [4] and the results are shown below in Figure 4 and 5. In our calculations, we used $n_1 = n_2 = 42$ Chebyshev collocation points in each phase, and carried out the transient-growth calculations using only the first 10 eigenmodes. These choices were made on a trial-and-error basis: (n_1, n_2) were varied to obtain numerical convergence, and the cutoff of number of eigenmodes was varied so as to find maximum agreement between the present calculations and those in the reference text. There are some discrepancies between the two approaches in Figure 4 while no such discrepancies are in evidence in Figure 5. The small discrepancies are not concerning, since the results depend slightly on the cutoff number of eigenmodes in the calculations, and this is not stated explicitly in the reference text. However, the excellent agreement between the two approaches in Figure 5 reinforces our confidence in our own implementation of the transient-growth calculation.

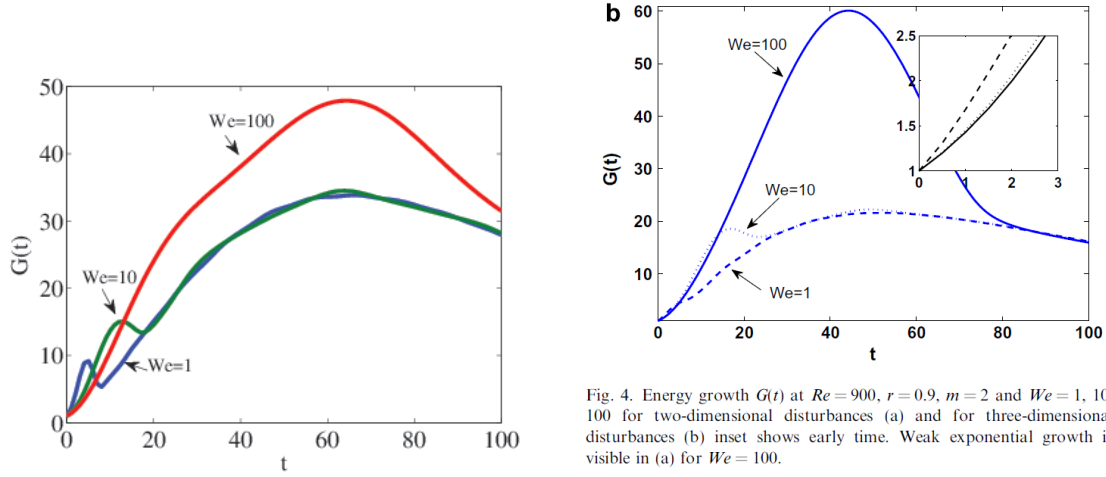


FIG. 4. Comparison with Figure 4 in Reference [4]: panel (a) this work; panel (b): taken directly from the reference text

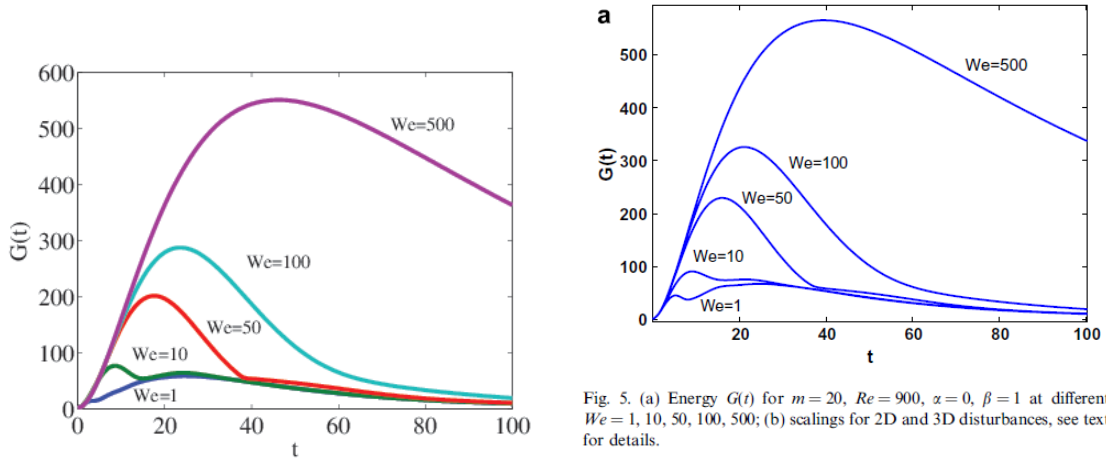


FIG. 5. Comparison with Figure 5 in Reference [4]: panel (a) this work; panel (b): taken directly from the reference text

IV. THE RESOLVENT

The resolvent of the evolutionary equation in the energy norm can also be computed along these lines. By definition,

$$R(z) = \|(\mathcal{L}_\alpha - z\mathcal{M}_\alpha)^{-1}\|_E, \quad (17)$$

with $R(z) = \infty$ when $z \in \text{spec}(\mathcal{L}_\alpha, \mathcal{M}_\alpha)$. Introduce $T = \mathbb{T}/2k^2$ and $\mathcal{R}(z) = (\mathcal{L}_\alpha - z\mathcal{M}_\alpha)^{-1}$. Using the definition (17) and the definition of the operator norm, it should be clear that Equation (17) can be rewritten as

$$[R(z)]^2 = \sup_{\langle \mathbf{x}, T\mathbf{x} \rangle = 1} \langle \mathcal{R}(z)\mathbf{x}, T\mathcal{R}(z)\mathbf{x} \rangle, \quad (18)$$

where the angle brackets denote the usual scalar product. We introduce a constrained functional,

$$F[\mathbf{x}] = \langle \mathcal{R}(z)\mathbf{x}, T\mathcal{R}(z)\mathbf{x} \rangle - \lambda (\langle \mathbf{x}, T\mathbf{x} \rangle - 1),$$

such that $\delta F/\delta \mathbf{x}^* = 0$ solves the optimization problem (18), hence

$$\mathcal{R}(z)^\dagger T\mathcal{R}(z)\mathbf{x} = \lambda T\mathbf{x}, \quad (19)$$

hence

$$[R(z)]^2 = \max \{ \text{spec} [\mathcal{R}(z)^\dagger T\mathcal{R}(z), T] \},$$

where $\text{spec} [\mathcal{R}(z)^\dagger T\mathcal{R}(z), T]$ denotes the spectrum of the generalized eigenvalue problem (19).

Downloads

The single-phase and two-phase transient growth calculators used in this Report can be downloaded:

<http://mathsci.ucd.ie/~onaraigh/tpls.html>

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